Proofs from THE BOOK

Reviewed by Daniel H. Ullman

Proofs from THE BOOK
Martin Aigner and *Günter M. Ziegler*
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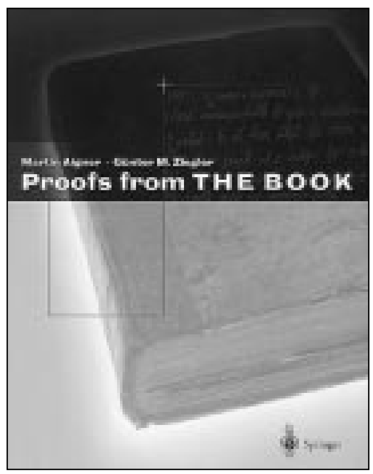
"You don't have to believe in God, but you have to believe in *The Book*."—Paul Erdős

Is mathematics a religion? This question first occurred to me when I attended a session dedicated to the memory of Paul Erdős at the San Diego meetings in January 1998. There, many people spoke about the life of this man of legend, and this is what I heard: Erdős was a priest of mathematics, singularly devoted to this one passion. He traveled far and wide sharing his form of gospel. And he believed that his purpose on Earth was to "conjecture and prove." Many people in the outside world view this dedication as a peculiarity; they may admire the genius but cannot comprehend the mission. But we in the mathematics community share his faith in the meaningfulness of mathematics. We on the inside speak the same language, practice the same rituals, seek the same goals. We are therefore in a unique position to appreciate the greatness and the goodness of the man. We are in the fold. We are the believers.

Even while he was alive, the legend of Erdős was well established. The stories about his idiosyncracies are by now part of the folk culture of mathematics. Erdős was fond of referring to *The Book*, where the perfect proofs of all theorems are written. "This one is from *The Book*", he would intone when seeing a particularly beautiful argument. And he lived his life on a crusade to reveal and enjoy as much of *The Book* as possible.



Proofs from THE BOOK is an effort by Martin Aigner and Günter Ziegler to reveal an approximation to a portion of *The Book*. (Let us denote by PFTB the book by Aigner and Ziegler, so as not to confuse *The Book* with "the book".) They had hoped to publish PFTB on the occasion of Erdős's eighty-fifth birthday in March 1998, with Erdős as a coauthor. But Erdős died in September 1997, and so Aigner and Ziegler wrote PTFB themselves and dedicated it to his memory. It is in large part a tribute to the mathematical legacy of Erdős.

It is an ambitious undertaking. I found the title at first to be somewhat off-putting, since it seems to suggest unashamedly that the contents are ideal, perfect, impossible to improve. *The Book*, Erdős would have said, is in the possession of the S. F. (the Supreme Fascist, Erdős's name for the Almighty). It is not for us in this lifetime to know

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its secrets. What sort of prophets are Aigner and Ziegler that they could write such a book?

These initial reservations were swept aside as soon as I began reading. Inside PFTB is indeed a glimpse of mathematical heaven, where clever insights and beautiful ideas combine in astonishing and glorious ways. There is vast wealth within its pages, one gem after another. Some of the proofs are classics, but many are new and brilliant proofs of classical results. Still others are recent results. Here are four of my favorites:

1. Fred Galvin's proof of the Dinitz Conjecture. Suppose one is given, for each i and j with $1 \le i, j \le n$, a set $L(i, j)$ of size n. Is it possible to choose $c_{i,j} \in L_{i,j}$ in such a way that no row or column of the resulting n-by-n matrix C contains a repeated element? An affirmative answer to this question was known as the Dinitz Conjecture, a special case of what graph theorists call the List Coloring Conjecture. Galvin's proof from 1995 uses the idea of a stable matching, a concept much studied in recent decades because of practical applications such as matching residents to hospitals and matching applicants to colleges. Galvin's result is a triumph of combinatorical reasoning, turning applied mathematics on its head to advance the cause of pure mathematics.

2. Carsten Thomassen's proof of the Five-Color Theorem. The only known proofs of the Four-Color Theorem are so complicated that they can only be read with the assistance of computing machinery. (We are still waiting for the *Book* proof.) The Five-Color Theorem, by contrast, has a relatively simple proof. In fact, the ideas in Alfred Kempe's 1879 famous false proof of the Four-Color Theorem, namely Euler's formula and alternating chains, serve easily to prove the five-color result. But the *Book* proof of the Five-Color Theorem must certainly be Thomassen's 1994 proof, a delicate example of "induction loading". Thomassen avoids Euler's formula and alternating chains and obtains a substantial generalization of the Five-Color Theorem in a mere three paragraphs.

3. Don Zagier's proof of the characterization of numbers expressible as the sum of two squares. This is another example of a recent proof of a classical result. Zagier's paper, which appeared in the *American Mathematical Monthly* in 1990 with the title "A one-sentence proof that every prime $p \equiv 1 \pmod 4$ is a sum of two squares", proves the apparently stronger result that the number of such representations is odd. (In fact it is always 1.) While this result is sometimes attributed to Fermat, the first published proof is thought to be Euler's. The last published proof will no doubt be Zagier's.

4. H. Tverberg's proof of the optimal decomposition of a complete graph into complete bipartite graphs. The question here is into how few blocks can the edges of the complete graph K_n be partitioned if each block is the edge set of a complete bipartite graph? It is not difficult to find such a decomposition into $n - 1$ blocks, but the impossibility of decomposition into fewer than $n - 1$ blocks is more subtle. Ron Graham and Henry Pollak first proved this result using linear algebra in 1971. The proof of Tverberg appeared in 1982. No purely graph-theoretic proof is known.

In PFTB one can also find the best proof of Pick's Theorem, of the arithmetic mean–geometric mean inequality, and of the lemma of Littlewood and Offord. If you haven't seen it before, do not miss the beautiful solution to Sylvester's problem about finite sets in the plane. Or Erdős's proof of Bertrand's postulate, from his first publication. It goes on and on.

There are thirty short chapters organized into five general areas: number theory, geometry, analysis, combinatorics, and graph theory. Some of the chapters are organized around single theorems, such as Turán's Theorem, Borsuk's Conjecture, or Cayley's Formula. Others are essays on a collection of results, with titles such as "In praise of inequalities" or "Three famous theorems on finite sets". Throughout, the writing is polished, clean, simple, as such a book demands. I noted the influence of Erdős in over half of the chapters.

The authors intended to make their book "accessible to readers whose backgrounds include only a modest amount of technique from undergraduate mathematics." Yet the level of the exposition varies substantially. There is much here appropriate for the Olympiad-level high schooler or a talented undergraduate math major. But not many such students will have the sophistication to follow the arguments or the experience to appreciate their elegance. The margin of the first page contains a statement of Lagrange's Theorem on the size of a subgroup, together with a proof in four short sentences. The student who has never before seen Lagrange's Theorem is unlikely to follow this high-level proof and certainly will not be able to appreciate any of Chapter 5, entitled "Every finite division ring is a field". My guess is that this book will find the vast majority of its readers among professional mathematicians. Yet no such reader needs to have the notion of a bijection explained to them, as is done at some length in Chapter 16.

What qualifies a proof to be in *The Book*? What makes an argument beautiful? It isn't easy to explain to a nonmathematician. Perhaps it cannot be explained at all. One simply appreciates these things or one does not, and if one is the first type of person, one might want to be a mathematician. One cannot force a person to enjoy music or art. Neither can a person be made to feel anything when presented with a brilliant argument. In fact, numbness toward mathematics is more common among the general populace than numbness toward music or art. After all, while most people grow up hearing and seeing things, thus learning to combine sounds and sights (the elements of music and art), many people do not grow up with much experience in combining logical ideas. Such people are likely to develop a sort of tone deafness toward mathematics: they may possess an ability to listen to and perhaps even comprehend a mathematical argument, but no ability to appreciate one.

Even among mathematicians there will always be a debate about what is elegant, about what is clever, about what is beautiful. Aigner and Ziegler do not claim to have presented the definitive collection of great mathematics. In their brief introduction they write: "We have no definition or characterization of what constitutes a proof from *The Book*: all we offer is the examples that we have selected, hoping that our readers will share our enthusiasm about brilliant ideas, clever insights and wonderful observations." I do.

The only criterion I can discern from PTFB for what constitutes a brilliant or beautiful proof is that such a proof contains an unexpected combination of ideas. Erdős is perhaps best known for his invention of the probabilistic method in graph theory, introducing random variables to attack questions about nonrandom objects. The last chapter of PFTB is devoted to several gems stemming from this combination of ideas. The penultimate chapter is devoted to a proof of the so-called friendship theorem via linear algebra, where a purely graph-theoretic question is answered by analyzing eigenvalues. The first chapter contains a proof by Harry Furstenberg of the infinitude of primes using topology. This mixing of ideas is common to nearly every argument featured in PFTB. Music is not just notes; it is the way the notes are combined.

There are many reasons to do mathematics. Some mathematicians may labor to prove theorems for the betterment of humanity—to improve efficiency of business, say, or to yield new techniques for medical care. Some may seek only to build mathematics itself. Others may seek fame and fortune (misguided though they may be). Yet others explore mathematics just to be tickled by ideas. PFTB is for them. Although the book restricts itself to elementary mathematics, I doubt there are many mathematicians, no matter how seasoned, who will know all the proofs here. And for the rest of us, PFTB will reward our attention richly. In a mere 199 pages we can encounter many of the great theorems of elementary mathematics with their best-known proofs. And we'll be the wiser for it.