DE MOIVRE
ON THE LAW OF NORMAL PROBABILITY

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Abraham de Moivre (1667-1754) left France at the revocation of the Edict of Nantes and spent the rest of his life in London. where he solved problems for wealthy patrons and did private tutoring in mathematics. He is best known for his work on trigonometry, probability. and annuities. On November 12, 1733 he presented privately to some friends a brief paper of seven pages entitled "Approximatio ad Summam Terminorum Binomii $\overline{a+b}\backslash^n$ in Seriem expansi." Only two copies of this are known to be extant. His own translation, with some additions, was included in the second edition (1738) of *The Doctrine of Chances*, pages 235–243.

This paper gave the first statement of the formula for the "normal curve," the first method of finding the probability of the occurrence of an error of a given size when that error is expressed in terms of the variability of the distribution as a unit, and the first recognition of that value later termed the probable error. It shows, also, that before Stirling, De Molvre had been approaching a solution of the value of factorial n.

A Method of approximating the Sum of the Terms of the Binomial $\overline{a+b}\backslash^n$ *expanded to a Series from whence are deduced some practical Rules to estimate the Degree of Assent which is to be given to Experiments.*

Altho' the Solution of Problems of Chance often require that several Terms of the Binomial $\overline{a+b}\backslash^n$ be added together, nevertheless in very high Powers the thing appears so laborious, and of so great a difficulty, that few people have undertaken that Task; for besides *James* and *Nicolas Bernoulli*, two great Mathematicians, I know of no body that has attempted it; in which, tho' they have shewn very great skill, and have the praise which is due to their Industry, yet some things were farther required; for what they have done is not so much an Approximation as the determining very wide limits, within which they demonstrated that the Sum of the Terms was contained. Now the Method which they have followed has been briefly described in my *Miscellanea Analytica*, which the Reader may consult if he pleases, unless they rather chuse, which perhaps would he the best, to consult what they themselves have writ upon that Subject: for my part, what made me apply myself to that Inquiry was not out of opinion that I should excel others, in which however I might have been forgiven; but what I did was in compliance to the desire of a very worthy Gentleman, and good Mathematician, who encouraged me to it: I now add some new thoughts to the former; but in order to make their connexion the clearer, it is necessary for me to resume some few things that have been delivered by me a pretty while ago.

I. It is now a dozen years or more since I had found what follows; If the Binomial $1+1$ be raised to a very high Power denoted by n, the ratio which the middle Term has to the Sum of all the Terms, that is, to 2^n, may he expressed by the Fraction $\frac{2\text{A}\times\overline{n-1}\backslash^n}{n^n\times\sqrt{n-1}}$, wherein A represents the number of which the Hyperbolic Logarithm is $\frac{1}{12}-\frac{1}{360}+\frac{1}{1260}-\frac{1}{1680}$, &c. but because the Quantity $\frac{\overline{n-1}\backslash^n}{n^n}$ or $1-\frac{1}{n}\Big|^n$ is very nearly given when n is a high Power, which is not difficult to prove, it follows that, in an infinite Power, that Quantity will he absolutely given, and represent the number of which the Hyperbolic Logarithm is -1; from whence it follows, that if B denotes the Number of which the Hyperbolic Logarithm is $-1+\frac{1}{12}-\frac{1}{360}+\frac{1}{1260}-\frac{1}{1680}$, &c. the Expression above-written will become $\frac{2\text{B}}{\sqrt{n-1}}$ or barely $\frac{2\text{B}}{\sqrt{n}}$ and that therefore if we change the Signs of that Series, and now suppose that B represents the Number of which the Hyperbolic Logarithm is $1-\frac{1}{12}+\frac{1}{360}-\frac{1}{1260}+\frac{1}{1680}$, &c. that Expression will he changed into $\frac{2}{\text{B}\sqrt{n}}$.

When I first began that inquiry, I contented myself to determine at large the Value of B, which was done by the addition of some Terms of the above-written Series; but as I pereciv'd that it converged but slowly, and seeing at the same time that what I had done answered my purpose tolerably well, I desisted from proceeding farther, till my worthy and learned Friend Mr. *James Stirling*, who had applied himself after me to that inquiry, found that the Quantity B did denote the Square-root of the Circumference of a Circle whose Radius is Unity, so that if that Circumference he called c, the Ratio of the middle Term to the Sum of all the Terms will he expressed by $\frac{2}{\sqrt{nc}}$.[1]

But altho' it be not necessary to know what relation the number B may have to the Circumference of the Circle, provided its value be attained, either by pursuing the Logarithmic Series before mentioned, or any other way; yet I own with pleasure that this discovery, besides that it has saved trouble, has spread a singular Elegancy on the Solution.

II. I also found that the Logarithm of the Ratio which the middle Term of a high Power has to any Term distant from it by an Interval denoted by l, would he denoted by a very near approximation, (supposing

[1][Under the circumstances of De Moivre's problem, nc is equivalent to $8\sigma^2\pi$, where σ is the standard deviation of the curve. This statement therefore shows that De Moivre knew the maximum ordinate of the curve to be

$$y_0=\frac{1}{\sigma\sqrt{2\pi}}.]$$

$m = \frac{1}{2}n$) by the Quantities $\overline{m + l - \frac{1}{2}} \times \log .\overline{m + l - 1} + \overline{m - l + \frac{1}{2}} \times \overline{\log .m - l + 1} - 2m \times \log .m + \log .\frac{m+l}{m}$.

Corollary I.

This being admitted, I conclude, that if m or $\frac{1}{2}n$ be a Quantity infinitely great. then the logarithm of the Ratio, which a Term distant from the middle by the Interval l, has to the middle Term, is $-\frac{2ll}{n}$.[2]

Corollary 2.

The Number, which answers to the Hyperbolic Logarithm $-\frac{2ll}{n}$, being

$$1 - \frac{2ll}{n} + \frac{4l^4}{2nn} - \frac{8l^6}{24n^4} - \frac{32l^{10}}{120n^5} + \frac{64l^{12}}{720n^6}, \text{ \&c.}$$

it follows, that the Sum of the Terms intercepted between the Middle, and that whose distance from it is denoted by L, will be $\frac{2}{\sqrt{nc}}$ into $l - \frac{2l^3}{1\times 3n} + \frac{4l^5}{2\times 5nn} - \frac{8l^7}{6\times 7n^3} + \frac{16l^9}{24\times 9n^4} - \frac{32l^{11}}{120\times 11n^5}$, &c.

Let now l be supposed $= s\sqrt{n}$, then the said Sum will be expressed by the Series

$\frac{2}{\sqrt{nc}}$ into $\int - \frac{2\int^3}{3} + \frac{4\int^5}{2\times 5} - \frac{8\int^7}{6\times 7} - \frac{16\int^9}{24\times 9} - 32\frac{\int^{11}}{120\times 11}$, &c.[3]

Moreover, if $\int$ be interpreted by $\frac{1}{2}$, then the Series will become $\frac{2}{\sqrt{c}}$ into $\frac{1}{2} - \frac{1}{3\times 4} + \frac{1}{2\times 5\times 8} - \frac{1}{6\times 7\times 16} + \frac{1}{24\times 9\times 32} + \frac{1}{120\times 11\times 64}$, &c. which converges so fast, that by help of no more than seven or eight Terms, the Sum required may he carried to six or seven places of Decimals: Now that Sum will he found to be 0.427812, independently from the common Multiplicator $\frac{2}{\sqrt{c}}$, and therefore to be the Tabular Logarithm[4] of 0.427812, which is $\overline{9}.6312529$, adding the Logarithm of $\frac{2}{\sqrt{c}}$ *viz.* $\overline{9}.9019400$, the Sum will be $\overline{19}.5331929$, to which answers the number 0.341344.

[2][Since $n = 4\sigma^2$ under the assumptions made here, this is equivalent to stating the formula for the curve as

$$y = y_0 \exp^{-\frac{x^2}{2\sigma^2}}.\Big]$$

[3][The long $\int$ which De Moivre employed in this formula is not to be mistaken for the integral sign.]
[4][to base 10.]

Lemma

If an Event be so dependent on Chance, as that the Probabilities of its happening or failing be equal, and that a certain given number n of Experiments be taken to observe how often it happens and fails, and also that l be another given number, less than $\frac{1}{2}n$. then the Probability of its neither happening more frequently than $\frac{1}{2}n + l$ times, nor more rarely than $\frac{1}{2}n - l$ times, mav he found as follows.

Let L and L be two Terms equally distant on both sides of the middle Term of the Binomial $\overline{1+1}\backslash^n$ expanded, by an Interval equal to l; let also $\int$ be the Sum of the Terms included between L and L together with the Extreme, then the Probability required will he rightly expressed by the Fraction $\frac{\int}{2^n}$, which being founded on the common Principles of the Doctrine of Chances, requires no Demonstration in this place.

Corollary 3.

And therefore, if it was possible to take an infinite number of Experiments, the Probability that an Event which has an equal number of Chances to happen or fail, shall neither appear more frequently than in $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times, nor more rarely than in $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ times, will he express'd by the double Sum of the number exhibited in the second Corollary, that is, by 0.682688, and consequently the Probability of the contrary, which is that of happening more frequently or more rarely than in the proportion above assigned will he 0.317312. these two Probabilities together compleating Unity, which is the measure of Certainty: Now the Ratio of those Probabilities is in small Terms 28 to 13 very near.

Corollary 4.

But altho' the taking an infinite number of Experiments he not practicable, yet the preceding Conclusions may very well he applied to finite numbers, provided they he great, for Instance, if 3600 Experiments he taken, make $n = 3600$, hence $\frac{1}{2}n$ will be = 1800, and $\frac{1}{2}\sqrt{n}$ 30, then the Probability of the Event's neither appearing oftner than 1830 times, nor more rarely than 1770, will he 0.682688.

Corollary 5.

And therefore we may lay this down for a fundamental Maxim, that in high Powers, the Ratio, which the Sum of the Terms included between two Extreams distant on both sides from the middle Term by an Interval equal to $\frac{1}{2}\sqrt{n}$, bears to the Sum of all the Terms, will he rightly expres'd by the Decimal 0.682688, that is $\frac{28}{41}$ nearly.

Still, it is not to be imagin'd that there is any necessity that the number n should he Immensely great; for supposing it not to reach beyond the 900^{th}, Power, nay not even beyond the 100^{th}, the Rule here given will be tolerably accurate, which I have had confirmed by Trials.

But it is worth whole to observe, that such a small part as is $\frac{1}{2}\sqrt{n}$ in respect to n, and so much the less in respect to n as n increases, does very soon give the Probability $\frac{28}{41}$ or the Odds of 28 to 13; from whence we may naturally he led to enquire, what are the Bounds within which the proportion of Equality is contained; I answer, that these Bounds will he set at such a distance from the middle Term, as will be expressed by $\frac{1}{4}\sqrt{2n}$ very near; so in the case above mentioned, wherein n was supposed $= 3600$, $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times, nor more rarely than in $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ times, will he express'd by the double Sum of the number exhibited in the second Corollary, that is, by 0.682688, and consequently the Probability of the contrary, which is that of happening more frequently or more rarely than in the proportion above assigned will he 0.317312. these two Probabilities together compleating Unity, which is the measure of Certainty: Now the Ratio of those Probabilities is in small Terms 28 to 13 very near.

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Corollary 6.

If l he interpreted by $\sqrt{n}$, the Series will not converge so fast as it did in the former case when l was interpreted by $\frac{1}{2}\sqrt{n}$, for here no less than 12 or 13 Terms of the Series will afford a tolerable approximation, and it would still require more Terms, according as l bears a greater proportion to $\sqrt{n}$, for which reason I make use in this case of the Artifice of Mechanic Quadratures, first invented by Sir *Isaac Newton*, and since prosecuted by Mr. *Cotes*. Mr. *James Stirling*, myself, and perhaps others; it consists in determining the Area of a Curve nearly, from knowing a certain number of its Ordinates A, B, C, D, E, F, &c. placed at equal Intervals, the more Ordinates there are, the more exact will the Quadrature be; but here I confine myself to four, as being sufficient for my purpose: let us therefore suppose that the four Ordinates are A, B, C, D, and that the Distance between the first and last is denoted by l, then the Area contained between the first and the last will be $\frac{1\times\overline{A+D}+3\times\overline{B+C}}{8}\times l$; now let us take the Distances $0\sqrt{n}$, $\frac{1}{6}\sqrt{n}$, $\frac{2}{6}\sqrt{n}$, $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, of which every one exceeds the preceding by $\frac{1}{6}\sqrt{n}$, and of which the last is $\sqrt{n}$; of these let us take the four last, *viz.* $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, then taking their Squares, doubling each of them, dividing them all by n. and prefixing to them all the Sign $-$, we shall have $-\frac{1}{2}$ $-\frac{8}{9}$, $-\frac{1}{2}$, $-\frac{25}{18}$, $-\frac{2}{1}$, which must he look'd upon as Hyperbolic logarithms, of which consequently the corresponding numbers, viz. 0.60653, 0.41111, 0.24935. 0.13534 will stand for the four Ordinates A, B, C, D. Now having interpreted l by $\frac{1}{2}\sqrt{n}$, the Area will be found to be $= 0.170203 \times \sqrt{n}$, the double of which being multiplied by the product will he 0.27160; let therefore this he added to the Area found before, that is, to 0.682688, and the Sum 0.95428 will shew, what after a number of Trials denoted by n, the Probability will he of the Event's neither happening oftner than $\frac{1}{2}n+\sqrt{n}$ times, nor more rarely than $\frac{1}{2}n+\sqrt{n}$, and therefore the Probability of the contrary will be 0.04572. which shews that the Odds of the Event's neither happening oftner nor more rarely than within the Limits assigned are 21 to 1 nearly.

And by the same way of reasoning, it will he found that the probability of the Event's neither appearing oftner $\frac{1}{2}n+\frac{3}{2}\sqrt{n}$ nor more rarely than $\frac{1}{2}n-\frac{3}{2}\sqrt{n}$ will he 0.99874, which will make it that the Odds in this case will he 369 to 1 nearly.

To apply this to particular Examples, it will he necessary to estimate the frequency of an Event's happening or failing by the Square-root of the number which denotes how many Experiments have been, or are designed to be taken, and this Square-root, according as at has been already hinted at in the fourth Corollary, will he as it were the Modulus by which we are to regulate our Estimation, and therefore suppose the number of Experiments to be taken is 3600, and that it were required to assign the Probability of the Event's neither happening oftner than 2850 times, nor more rarely than 1750, which two numbers may be varied at pleasure, provided they he equally distant from the middle Sum 1800, then make the half difference between the two numbers 1850 and 1750, that is, in this case, $50 = \int\sqrt{n}$; now having supposed $3600 = n$, then $\sqrt{n}$ will be 60, which wdl make it that 50 will be $= 60\int$, and consequently $\int = \frac{50}{60} = \frac{5}{6}$, and therefore if we take the proportion, which in an infinite power, the double Sum of the Terms corresponding to the Interval $\frac{5}{6}\sqrt{n}$, bears to the Sum of all the Terms, we shall have the Probability required exceeding near.

Lemma 2.

In any Power $\overline{a+b}\backslash^n$ expanded. the greatest Term is that in which the Indices of the Powers of a and b. have the same proportion to one another as the Quantities themselves a and b; thus taking the 10^{th} Power of $a+b$, which is $a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}$ and supposing that the proportion of a to b is as 3 to 2, then the Term $210a^6b^4$ will be the greatest, by reason that the Indices of the Powers of a and b, which are in that Term, are in the proportion of 3 to 2; but supposing the proportion of a to b had been as 4 to 1, then the Term $45a^8b^2$ had been the greaust

Lemma 3.

If an Event so depends on Chance, as that the Probabilities of its happening or failing be in any assigned proportion, such as may he supposed of a to b, and a certain number of Experiments be designed to be taken, in order to observe how often the Event will happen or fail; then the Probability that it shall neither happen more frequently than so many times as are denoted by $\frac{an}{a+b}+l$, nor more rarely than so many times as are denoted by $\frac{an}{a+b}-l$ will be found as follows:

Let L and R be equally distant by the Interval l from the greatest Term; let also S be the Sum of the Terms included between L and R, together with those Extreams, then the Probability required will be rightly expressed by $\frac{\mathrm{S}}{\overline{a+b}\backslash^n}$.

Corollary 8.[5]

The Ratio which, in an infinite Power denoted by n. the greatest Term bears to the Sum of all the rest, will be rightly expressed by the Fraction $\frac{a+b}{\sqrt{abnc}}$, wherein c denotes, as before, the Circumference of a Circle for a Radius equal to Unity.

[5] [Numbered as in the original. There is no corollary 7 in the text.]

Corollary 9.

If, in an infinite Power. any Term be distant from the Greatest by the Interval l, then the hyperbolic logarithm of the Ratio which that Term bears to that Greatest will be expressed by the Fraction $-\frac{\overline{a+b}^2}{2abn} \times ll$; provided the Ratio of l to n be not a finite Ratio, but such a one as may he conceived between any given number p and $\sqrt{n}$, so that l be expressible by $p\sqrt{n}$, in which case the two Terms L and R will be equal.

Corollary 10.

If the Probabilities of happening and failing be in any given Ratio of unequality, the Problems relating to the Sum of the Terms of the Binomial $\overline{a+b}\backslash^n$ will be solved with the same facility as those in which the Probabilities of happening and failing are in a Ratio of Equality.

From what has been said, it follows, that Chance very little disturbs the Events which in their natural Institution were designed to happen or fail, according to some determinate law; for if in order to help our conception, we imagine a round piece of Metal, with two polished opposite faces, differing in nothing but their colour, whereofore may be supposed to be white, and the other black; it is plain that we may say, that this piece may with equal facility exhibit a white or black face, and we may even suppose that it was framed with that particular view of shewing sometimes one face, sometimes the other, and that consequently If it be tossed up Chance shall decide the appearance; but we have seen in our LXXXVII^th Problem, that altho' Chance may produce an inequality of appearance. and still a greater inequality according to the length of time in which it may exert itself, still the appearances, either one way or the other, will perpetually tend to a proportion of Equality; but besides we have seen in the present Problem, that In a great number of Experiments, such as 3600, it would be the Odds of above 2 to 1, that one of the Faces, suppose the white, shall not appear more frequently than 1830 times, nor more rarely than 1770, or in other Terms, that it shall not be above or under the perfect Equality by more than $\frac{1}{120}$ part of the whole number of appearances; and by the same Rule, that if the number of Trials had been 14400 instead of 3600, then still it would be above the Odds of 2 to 1, that the appearances either one way or other would not deviate from perfect Equality by more than $\frac{1}{260}$ part of the whole, but in 1000000 Trials it would be the Odds of above 2 to 1, that the deviation from perfect Equality would not he more than by $\frac{1}{2000}$ part of the whole. But the Odds would increase at a prodigious rate, if instead of taking such narrow limits on both sides the Term of Equality, as are represented by $\frac{1}{2}\sqrt{n}$, we double those Limits or triple them; for in the first case the Odds would become 21 to 1, and in the second 369 to 1, and still he vastly greater if we were to quadruple them, and at last be infinitely great; and yet whether we double, triple or quadruple them, &c. the Extension of those Limits will bear but an inconsiderable proportion to the whole, and none at all, if the whole be infinite, of which the reason will easily be perceived by Mathematicians, who know, that the Square-root of any Power bears so much a less proportion to that Power, as the Index of it is great.

And what we have said is also applicable to a Ratio of Inequality, as appears from our 9th Corollary. And thus in all cases it will he found, that altho' Chance produces irregularities, still the Odds will be infinitely great, that in process of Time, those Irregularities will bear no proportion to the recurrency of that Order which naturally results from original Design.